ON ONE FAMILY OF 13-DIMENSIONAL CLOSED RIEMANNIAN POSITIVELY CURVED MANIFOLDS

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1. Introduction and main results

In the present paper we describe one family of closed Riemannian manifolds with positive sectional curvature.

Now the list of known examples is not large (for instance, all known manifolds with dimension > 24 are diffeomorphic to compact rank one symmetric spaces) (we restrict ourselves only by pointing out simply connected manifolds) :

1) Berger described all normally homogeneous closed positively curved manifolds that are compact rank one symmetric spaces (i.e., the spheres S^n, the complex projective spaces CP^n, the quaternionic projective spaces HP^n, and the projective Cayley plane CaP^2), and two exceptional spaces of form $Sp(2)/SU(2)$ and $SU(5)/Sp(2) \times S^1$ with dimension 7 and 13, respectively (notice that the embedding $SU(2) \subset Sp(2)$ is not standard) ([Be]);

2) Wallach had shown that all even-dimensional simply connected closed positively curved are diffeomorphic to normally homogeneous ones or the flag spaces over CP^2, HP^2, and CaP^2 (with dimension 6,12, and 24, respectively) ([W]);

3) Aloff and Wallach ([AW]) constructed infinite series of spaces $N_{p,q}$ of the form $SU(3)/S^1$ where the subgroup S^1 is a winding of a maximal torus of group $SU(3)$ and, since that, is defined by a pair of relatively prime integer parameters p and q. If some conditions for these parameters p and q hold then these manifolds admit left-invariant homogeneous Riemannian metric with positive sectional curvature. Berard-Bergery ([BB]) had shown that the Aloff-Wallach spaces are all possible manifolds that admit homogeneous positively curved metric and do not admit normally homogeneous one, and Kreck and Stolz had found among them a pair of homeomorphic but nondiffeomorphic manifolds ($N_{-56788,5227}$ and $N_{-42652,61213}$; [KS]) ;

4) by using of the construction of Aloff and Wallach, Eschenburg had found an infinite series of seven-dimensional spaces with nonhomogeneous positively curved metrics ([E1]) and in the sequel had found an example of six-dimensional nonhomogeneous space with positively curved metric ([E2]).

This list contains all known up to now topological types of simply connected closed manifolds that admit metrics with positive sectional curvature. Notice that only two of them have dimension 13 : the sphere S^{13} and the normally homogeneous Berger space $SU(5)/Sp(2) \times S^1$.

The main result of the present paper is the construction of the new series of simply connected closed 13-dimensional manifolds that admit positively curved metrics. In particular, we prove the following theorem.

Main Theorem. *Let $U(5)$ be a group of complex unitary 5×5-matrices and a group $U(4) \times U(1)$ is embedded into it as a subgroup of matrices of block form with two blocks with size 4×4 and 1×1. Let M^{25} be a homogeneous Riemannian manifold diffeomorphic to $U(5)$ and endowed by metric induced from two-sided invariant metric on $U(5) \times U(4) \times U(1)$ by projection*

$$U(5) \times U(4) \times U(1) \to U(5) \times U(4) \times U(1)/U(4) \times U(1) = M^{25}$$

with diagonal embedding $U(4) \times U(1) \to U(5) \times U(4) \times U$ $(g \to (g, g) \in U(5) \times (U(4) \times U(1)))$.

Let $\bar{p} = (p_1, \ldots, p_5)$ be a 5-tupel of integer numbers such that for every permutation $\sigma \in S_5$ the following conditions hold

a) $p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)} \quad p_{\sigma(5)}$,

b) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} > p_{\sigma(4)} + p_{\sigma(5)}$,

c) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} + p_{\sigma(4)} > 3p_{\sigma(5)}$,

d) $3(p_{\sigma(1)} + p_{\sigma(2)}) > p_{\sigma(3)} + p_{\sigma(4)} + p_{\sigma(5)}$.

Let $M_{\bar{p}}$ be a factor-space $M_{\bar{p}}$ of M^{25} under the action of $S^1 \times (Sp(2) \times S^1)$ given by

$$(z_1, (A, z_2)) : X \to diag(z_1^{p_1}, z_1^{p_2}, z_1^{p_3}, z_1^{p_4}, z_1^{p_5}) \cdot X \cdot \begin{pmatrix} A^* \bar{z}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $X \in M^{25}$, $z_1, z_2 \in S^1$, and $A \in Sp(2)$. Let $M_{\bar{p}}$ is endowed by metric induced by factorization $M^{25} \to M_{\bar{p}}$ then

1) $M_{\bar{p}}$ is simply connected and $\dim M_{\bar{p}} = 13$;

2) $M_{\bar{p}}$ has positive sectional curvature ;

3) the groups of cohomologies of $M_{\bar{p}}$ are the following ones :

$$H^i = \begin{cases} \mathbf{Z}, & for \quad i=0,2,4,9,11,13, \\ 0, & for \quad i=1,3,5,7,10,12; \end{cases}$$

the groups H^6 and H^8 are finite and their orders are equal to $|\sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3|$ where σ_k is a value of an elementary symmetric polynom, of degree k, on five variables at $(p_1, \ldots, p_5)$.

Remarks. The condition a)-d) hold, for instance, for $p_1 = 1, p_2 = p_3 = p_4 = p_5 = q^n$ where q is a prime number. In this case the order of $H^6(M_{\bar{p}})$ is equal to $r(q,n) = 8q^{2n} - 4q^n + 1$ and $r(q,n) \to \infty$ as $q \to \infty$. It follows from this example that there exists infinitely many pairwise nonhomeomorphic closed simply connected positively curved manifolds of the form $M_{\bar{p}}$.

One can see that for $n = 0$ we obtain manifold which is diffeomorphic to the 13-dimensional Berger space.

There exist another series and the simplest construction of them was pointed out to us by U. Abresch. In particular, let take a 5-tupel of numbers for which the condition a) of Main Theorem holds (notice here that we call two numbers relatively prime if their maximal common divisor is equal to one) and no one of numbers $|p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)}|$ vanishes. Let add to $p_1, \dots, p_5$ the same natural number $a_n = n \cdot \prod_{\sigma \in S_5} |p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)}|$. One can see that there exists sufficiently large number N such that for all $n > N$ 5-tupels $(p_1 + a_n, \dots, p_5 + a_n)$ satisfy to conditions b)-d) and it is easy to observe that these tupels always satisfy condition a). For instance, one can start from initial 5-tupel of the form $(1, 1, 1, 2q, 4q)$ where q is a prime number.

General structure of the family of constructed (in Theorem 1) manifolds and pinchings of their metrics will be considered separately.

At the construction of metric we follow to methods developed in [E1]. But at the proving of positivity of curvature the method introduced in [E1] met some difficulties which are passed by using of Lemma 8.

In the next chapter the spaces $M_{\bar{p}}$ are constructed, the statement 2 of the theorem is proved in chapter 3 (Theorem 1), and statements 1 and 3 are proved in the fourth, final, chapter (Theorems 2 and 3, respectively).

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2. Construction of spaces $M_{\bar{p}}$

2.1. Riemannian submersion and it's properties.

Let M and N be Riemannian manifolds and $f : M \to N$ be a smooth mapping. A mapping f is called *submersion* if f is surjective (i.e., $f(M) = N$) and the linear mapping $d_x : T_xM \to T_{f(x)}N$ is isomorphism for every point $x \in M$. Then at every point $x \in M$ the tangent space to M canonically decomposes into a direct sum of two subspaces $T_xM = (T_xM)^v \oplus (T_xM)^h$ where

$$(T_xM)^v = T_xK, \quad K = f^{-1}(f(x))$$

and $(T_xM)^h$ is an orthogonal complement to $(T_xM)^v$. These subspaces are called vertical and horizontal, respectively. It is evident that $d_xf|_{(T_xM)^h} : (T_xM)^h \to T_{f(x)}N$ is an isomorphism. If this isomorphism preserves metric the mapping f called *Riemannian submersion.*

The next lemma gives a general construction of examples of Riemannian submersions.

Lemma 1. *Let G be a group of isometries that acts freely and with closed orbits on a Riemannian manifold M. Then on the space of orbits one can introduce the*

structure of Riemannian manifold such that the natural projection $\pi : M \to N$ *be a Riemannian submersion.*

For Riemannian submersions the curvatures of manifolds M and N are related by formula found in [ON]. We restrict ourselves only by it's corrolary that we will need in the sequel.

Lemma 2. *Let* $\pi : M \to N$ *be a Riemannian submersion. Put* $x \in M, y \in N, \pi(x) = y$. *If* σ^* *is a two-dimensional horizontal plane in* T_xM *and* $\sigma = d_x\pi(\sigma^*)$ *then*

$$K(\sigma) \geq K(\sigma^*).$$

The proof of the next lemma one can find, for instance, in [M].

Lemma 3. *Let* G *be a Lie group with two-sided invariant metric* $\langle\ ,\ \rangle$ *and* $\mathbf{g}$ *be a tangent space, at the unit, endowed by a structure of Lie algebra. Then for any* $X, Y \in \mathbf{g}$ *a sectional curvature in direction* $Span(X, Y)$ *is equal to*

$$K(X, Y) = \frac{1}{4}\langle [X, Y], [X, Y]\rangle$$

2.2. Normally homogeneous metric on $U(5)$.

In this subchapter we construct one Riemannian metric on $U(5)$ and in the next subchapter we define free actions of the group $S^1 \times (Sp(2) \times S^1)/Z_2$ on $U(5)$ that are isometries with respect to this metric. This construction of an auxiliary metric on $U(5)$ gives itself an example of Riemannian submersion. Moreover metrics looked for on the spaces of orbits, of action $S^1 \times (Sp(2) \times S^1)/Z_2$ on $U(5)$, will be constructed by this metric with using of Lemma 1. At this construction we follow paper [E2].

Let G be the Lie group $U(5)$ and $K = U(4) \times U(1)$ be the subgroup which is embedded in the standard manner. Let consider a usual two-sided invariant Riemannian metric $\langle\ ,\ \rangle_0$ on G :

$$\langle X, Y\rangle_0 = Re \ \ trace \ \ (XY^*)$$

where $X, Y \in \mathbf{u}(5)$. This metric canonically induces metrics on K and $G \times K$. These metrics we will also define by $\langle\ ,\ \rangle_0$.

Let $\triangle K = \{(k, k)|k \in K\}$ be a subgroup in $G \times K$. We consider an action $\triangle K$ on $G \times K$ by right shifts :

$$((g, k), k') \longmapsto (gk', kk')$$

for $g \in G, k, k' \in K$. It is evident that that is an isometrical free action. By Lemma 1, there exists a metric on the space of orbits $(G \times K)/\triangle K$ such that the natural projection

$$\pi : G \times K \to (G \times K)/\triangle K$$

is a Riemannian submersion. One can see that the correspondence $(g,k) \to gk^{-1}$ gives a diffeomorphism $(G \times K)/\triangle K$ with G. By pulling, with the help of this diffeomorphism, the Riemannian metric from the space of orbits $(G \times K)/\triangle K$ onto G we obtain a metric $\langle\ ,\ \rangle$ on G. For that the mapping

$$\pi : G \times K \to G : (g,k) \longmapsto gk^{-1}$$

is a Riemannian submersion.

Let consider a left shift by element (g, k^{-1}) on the group $G \times K$ where $g \in G, k \in K$. Since the metric $\langle\ ,\ \rangle_0$ is two-sided invariant, this mapping is an isometry. Moreover, the left shift maps fibers of submersion into fibers and hence induces a mapping

$$g' \longmapsto gg'k : G \to G,$$

on G, that is an isometry. Thus, we conclude that *the metric $\langle\ ,\ \rangle$ is left-invariant under G and right-invariant under K.*

Let $\mathbf{k} = \mathbf{u}(4) \oplus \mathbf{u}(1)$ and $\mathbf{g} = \mathbf{u}(5)$ be tangent algebras of groups G and K , respectively. We denote by $\mathbf{p}$ an orthogonal complement to $\mathbf{k}$ in $\mathbf{g}$ with respect to the metric $\langle\ ,\ \rangle_0$. Then the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ is invariant under $Ad(K)$. Moreover, G/K is the symmetric space CP^4 and, since that, we have

$$[\mathbf{k},\mathbf{k}] \subset \mathbf{k}, \quad [\mathbf{p},\mathbf{p}] \subset \mathbf{k}, \quad [\mathbf{k},\mathbf{p}] \subset \mathbf{p}. \tag{1}$$

The vertical subspace of submersion π at (e,e) is

$$V = \{(Z,Z) | Z \in \mathbf{k}\} = \triangle\mathbf{k}.$$

Hence $(X,Y) \in \mathbf{g} \oplus \mathbf{k}$ lies in the horizontal subspace H if

$$\langle (X,Y),(Z,Z)\rangle_0 = 0$$

for every $Z \in \mathbf{k}$ that implies

$$\langle X,Z\rangle_0 + \langle Y,Z\rangle_0 = 0,$$

$$\langle X+Y,Z\rangle_0 = 0$$

for every $Z \in \mathbf{k}$. Notice that in this case $X + Y \in \mathbf{p}$, i.e., $X_k + Y_k = 0$ and $Y = Y_k = -X_k$. We derive from that that

$$H = \{(X_k + X_p, -X_k) | X_k \in \mathbf{k},\ X_p \in \mathbf{p}\}$$

and $d_{(e,e)}\pi|_H : H \to \mathbf{g}$ is an isometry.

Since $d_{(e,e)}\pi(X,Y) = X - Y$, for every $X \in \mathbf{g}$ the following equality holds

$$(d_{(e,e)}\pi|_H)^{-1}(X) = (\frac{1}{2}X_k + X_p, -\frac{1}{2}X_k). \tag{2}$$

holds.

Lemma 4. *Let $X \in \mathbf{g}$ and $Y \in \mathbf{k}$. Then $\langle X, Y\rangle = \frac{1}{2}\langle X, Y\rangle_0$.*

Proof of Lemma 4.
By (2), we have

$$\langle X, Y\rangle = \langle(\frac{1}{2}X_k + X_p, -\frac{1}{2}X_k), (\frac{1}{2}Y_k + Y_p, -\frac{1}{2}Y_k)\rangle_0 =$$

$$= \langle\frac{1}{2}X_k + X_p, \frac{1}{2}Y\rangle_0 + \langle -\frac{1}{2}X_k, -\frac{1}{2}Y\rangle_0 = \langle X_k + X_p, \frac{1}{2}Y\rangle_0 = \frac{1}{2}\langle X, Y\rangle_0.$$

Lemma 4 is proved.

In the sequel we will mean by curvature of the space G it's curvature with respect to the metric $\langle\ ,\ \rangle$.

Lemma 5. *Let σ be a two-dimensional plane in $\mathbf{g}$ and $K(\sigma) = 0$. Then $\sigma = Span(X, Y)$, where $X \in \mathbf{g}$, $Y \in \mathbf{k}$ and*

$$[X_p, Y] = [X_k, Y] = 0$$

Proof of Lemma 5.
Let $\sigma = Span(X, Y)$ where $X, Y \in \mathbf{g}$. Let $\sigma^* = Span((\frac{1}{2}X_k + X_p, -\frac{1}{2}X_p), (\frac{1}{2}Y_k + Y_p, -\frac{1}{2}Y_k))$ lies in the horizontal subspace of submersion π. We have $d_e\pi(\sigma^*) = \sigma$. By Lemmas 2 and 3, $0 \le K(\sigma^*) \le K(\sigma) = 0$. That implies $K(\sigma^*) = 0$.

It follows from lemma 3 that

$$[(\frac{1}{2}X_k + X_p, -\frac{1}{2}X_k), (\frac{1}{2}Y_k + Y_p, -\frac{1}{2}Y_k)] = 0,$$

$$([\frac{1}{2}X_k + X_p, \frac{1}{2}Y_k + Y_p], [-\frac{1}{2}X_k, -\frac{1}{2}Y_k]) = 0.$$

Hence ,

$$[X_k, Y_k] = 0,$$

$$\frac{1}{2}[X_k, Y_p] + \frac{1}{2}[X_p, Y_k] + [X_p, Y_p] = 0$$

By (1), $[X_p, Y_p] \in \mathbf{k}$ and $[X_k, Y_p] + [X_p, Y_k] \in \mathbf{p}$, that implies

$$[X_p, Y_p] = 0,$$

$$[X_k, Y_p] + [X_p, Y_k] = 0.$$

Next, $X_p, Y_p \in \mathbf{p}$ are tangent vectors to positively curved space CP^4. Since the curvature of CP^4 in the direction $Span(X_p, Y_p)$ vanishes, vectors X_p, Y_p are linearly dependent. Hence, we may assume that $\sigma = Span(X, Y)$ where $Y \in \mathbf{k}$.

Then we obtain

$$[X_k, Y] = [X_p, Y] = 0.$$

Lemma 5 is proved.

2.3. Free actions on $U(5)$ and construction of spaces $M_{\bar{p}}$.

Let p_1, p_2, p_3, p_4, p_5 be integer numbers. Put $P' = S^1 \times (Sp(2) \times S^1)$ where we assume that $Sp(2)$ is standard embedded into $SU(4)$.

Let consider an action of the group P' on $G = U(5)$:

$$(z_1, (A, z_2)) : X \mapsto diag(z_1^{p_1}, z_1^{p_2}, z_1^{p_3}, z_1^{p_4}, z_1^{p_5}) \cdot X \cdot \begin{pmatrix} A^* \bar{z}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $X \in G$, $z_1, z_2 \in S^1$, $A \in Sp(2)$.

Lemma 6. *Let $p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)}$ is relatively prime with $p_{\sigma(5)}$ for every transposition $\sigma \in S_5$. Then this action has a kernel isomorphic to $Z_2 = (1, \pm(E, 1))$ and therefore induces a free action of group*

$$P = S^1 \times \frac{Sp(2) \times S^1}{\pm(E, 1)} =: P_1 \times P_2.$$

on G.

Proof of Lemma 6.

Let assume that

$$X = diag(z_1^{p_1}, z_1^{p_2}, z_1^{p_3}, z_1^{p_4}, z_1^{p_5}) \cdot X \cdot \begin{pmatrix} A^* \bar{z}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$diag(\bar{z}_1^{p_1}, \bar{z}_1^{p_2}, \bar{z}_1^{p_3}, \bar{z}_1^{p_4}, \bar{z}_1^{p_5}) = X \begin{pmatrix} A^* \bar{z}_2 & 0 \\ 0 & 1 \end{pmatrix} X^{-1}.$$

We consider the maximal torus in $Sp(2)$:

$$T^2 = \{diag(u, v, \bar{u}, \bar{v}) | u, v \in S^1\}.$$

Then there exists an element $Y \in Sp(2)$ such that $A^* = Y diag(u, v, \bar{u}, \bar{v}) Y^{-1}$ for some $u, v \in S^1$. So, we have

$$diag(\bar{z}_1^{p_1}, \bar{z}_1^{p_2}, \bar{z}_1^{p_3}, \bar{z}_1^{p_4}, \bar{z}_1^{p_5}) =$$

$$= \left(X \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \right) diag(u\bar{z}_2, v\bar{z}_2, \bar{u}\bar{z}_2, \bar{v}\bar{z}_2, 1) \left(X \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1}.$$

That means that there exist a permutation i such that $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$ and

$$\bar{z}_1^{p_{i_1}} = u\bar{z}_2, \quad \bar{z}_1^{p_{i_2}} = v\bar{z}_2, \quad \bar{z}_1^{p_{i_3}} = \bar{u}\bar{z}_2, \quad \bar{z}_1^{p_{i_4}} = \bar{v}\bar{z}_2, \quad \bar{z}_1^{p_{i_5}} = 1.$$

It follow from the first four equalities that

$$\bar{z}_1^{p_{i_1} + p_{i_3} - p_{i_2} - p_{i_4}} = 1.$$

By the condition of Lemma, p_{i_5} is relatively prime with $p_{i_1}+p_{i_3}-p_{i_2}-p_{i_4}$, and therefore $\bar{z}_1=1$, i.e., $z_1=1$. Then $\bar{z}_2^2=1$, $z_2=\pm 1$.

1) If $z_2=1$ then $u=v=1, A^*=YEY^{-1}=E$, i.e., $A=E$.

2) If $z_2=-1$ then $u=v=-1, A=-E$.

Thus, the kernel of action is given by

$$Z_2=(1,\pm(E,1)).$$

Lemma 6 is proved.

The group P acts on G by isometries. Therefore, by Lemma 1, one can introduce a Riemannian structure of the space of orbits $M_{\bar{p}}$ such that

$$\bar{\pi}: G \to M_{\bar{p}}$$

is an isometry.

3. Curvature of spaces $M_{\bar{p}}$

In this chapter we will find conditions on $\bar{p}$ under that sectional curvature of $M_{\bar{p}}$ is positive.

The following Lemma 7 was proved in [E2] but for the sake of completeness of explanation we give it's proof.

Lemma 7. *Let G be a compact Lie group with two-sided invariant metric $\langle\ ,\ \rangle_0$ and $\mathbf{t} \subset \mathbf{g}$ be a maximal commutative subgroup in the tangent algebra to G. Let $H \in \mathbf{t}$. Put $M = Ad(G)A$ where $A \in \mathbf{g}$. Let consider a function*

$$f_H : M \to R : X \mapsto \langle H, X\rangle_0.$$

Then extremal values of f are attained on $M \cap \mathbf{t}$.

Proof of Lemma 7.

Firstly, let assume that an element H is regular, i.e., it is contained only in one maximal commutative subalgebra.

Let $X \in M$ be a critical point of f_H. That means that $d_X f_H = \langle H, -\rangle_0 = 0$. Since $M = Ad(G)X$, we have $T_X M = ad(\mathbf{g})X$. Therefore,

$$\langle ad(\mathbf{g})X, H\rangle_0 = 0$$

and

$$\langle [Z,X], H\rangle_0 = \langle Z, [X,H]\rangle_0 = 0$$

for every $Z \in \mathbf{g}$. Hence, $[X,H]=0$ and, by regularity of H, we conclude that $X \in \mathbf{t}$.

Let assume now that H is a singular element of $\mathbf{t}$. Let X an extremal point and $X \in \mathbf{g}\setminus\mathbf{t}$. By small perturbation of H we obtain a situation when H be a regular element and X still lies in $\mathbf{g}\setminus\mathbf{t}$.

Thus we arrive at a contradiction with statement proved before.

Lemma 7 is proved.

Lemma 8. *Let F be a subalgebra of* $\mathbf{s}u(4)$ *with dimension 10 and* $\mathbf{t}$ *be a maximal commutative subalgebra formed by diagonal matrices. Let assume that $H \in \mathbf{t}$ and $\langle H, F\rangle_0 = 0$. Then up to permutation there exist only two possibilities :*

$$H = i \cdot t \cdot diag(1, 1, -1, -1)$$

or

$$H = i \cdot t \cdot diag(1, 1, 1, -3)$$

where $t \in R$.

Proof of Lemma 8.

Let consider a transformation

$$ad(H) : \mathbf{s}u(4) \to \mathbf{s}u(4) : X \mapsto [H, X].$$

Take $X, Y \in F$. Then $[X, Y] \in F$, i.e., $\langle H, [X, Y]\rangle_0 = 0$. Therefore,

$$\langle [H, Y], X\rangle_0 = \langle H, [X, Y]\rangle_0 = 0.$$

Thus,

$$\langle F, ad\ H(F)\rangle_0 = 0$$

Moreover, $\langle [H, X], H\rangle_0 = \langle [H, H], X\rangle_0 = 0$ for every $X \in F$, i.e.,

$$\langle ad\ H(F), H\rangle_0 = 0.$$

Then

$$dim(ad\ H(F)) \leq dim\ \ \mathbf{s}u(4) - dimF - 1 = 4.$$

That means that

$$dim(Ker(ad\ H) \cap F) \geq 10 - 4 = 6.$$

Since $H \in Ker(ad\ H)$ and H does not lie in F, we have

$$dim(Ker(ad\ H)) \geq dim(Ker(ad\ H) \cap F) + 1 \geq 7.$$

Let consider the $ad\ H$-invariant root decomposition

$$\mathbf{s}u(4) = V_0 \oplus \bigoplus_{\substack{i,j=1 \\ i<j}}^{4} V_{i,j}$$

where $V_0 = \mathbf{t}, dimV_{i,j} = 2$. Then

$$ad\ H(V_0) = 0,$$

$$ad\ H(V_{i,j}) = \theta_{i,j}(H)V_{i,j}$$

where $\theta_{i,j}$ are roots of $\mathbf{s}u(4)$, i.e., $\theta_{i,j}(i \cdot diag(x_1, x_2, x_3, x_4)) = x_i - x_j$. By estimating of the dimension of the kernel of adH, we derive that at least two roots vanish at H.

Lemma 8 is proved.

Lemma 9. *Let for some $m \in M_{\bar{p}}$ there exists a two-dimensional plane $\sigma \subset T_m M_{\bar{p}}$ such that $K(\sigma) = 0$. Then there exist $g \in G$, $X, Y \in \mathbf{u}(5)$, such that X, Y are linearly independent, $K(X,Y) = 0$ and X, Y are orthogonal to a subspace*

$$D_g = \{Ad(g^{-1}) \cdot i \cdot t \cdot diag(p_1, p_2, p_3, p_4, p_5) - i \cdot s \cdot diag(1,1,1,1,0) - \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} |$$

$$|t, s \in R, A \in \mathbf{s}p(2)\}.$$

Proof of Lemma 9.

Let consider the Riemannian submersion

$$\bar{\pi} : G \to M_{\bar{p}}.$$

Put $\bar{\pi}(g) = m$ where $g \in G$. Then $T_g G = V \oplus H$ where V is a vertical subspace , H is a horizontal subspace, and $d_g \pi|_H$ is an isometry. We have

$$V = T_g P = \{d_e R_g(X) - d_e L_g(Y) | (X,Y) \in T_e P\},$$

where R_g and L_g are right and left shift by g , respectively. Then $H = V^{\perp}$ is an orthogonal complement.

Consequently, there exists $\sigma^* \in H$ such that $d_g \bar{\pi}(\sigma^*) = \sigma$. By Lemmas 2 and 3 , $0 \le K(\sigma^*) \le K(\sigma)$ and therefore $K(\sigma^*) = 0$. The left shift $L_{g^{-1}} = (L_g)^{-1}$ is an isometry. Let $d_g L_{g^{-1}}(\sigma^*) = Span(X,Y)$ where $X, Y \in T_e G = \mathbf{u}(5)$. Then $K(X,Y) = 0$ and $\langle V, \sigma^* \rangle = \langle d_g L_{g^{-1}}(V), Span(X,Y) \rangle = 0$.

Thus, X, Y are orthogonal to a subspace

$$D_g = d_g L_{g^{-1}}(V) =$$

$$= \{(d_g L_g)^{-1} (d_g R_g)(X) - Y | (X,Y) \in T_e P\} =$$

$$= \{d_e (L_{g^{-1}} \circ R_g)(X) - Y | (X,Y) \in T_e P\} =$$

$$= \{Ad(g^{-1}) X - Y | X \in T_e S^1, Y \in T_e(Sp(2) \times S^1)\} =$$

$$= \{Ad(g^{-1}) \cdot i \cdot t \cdot diag(p_1, p_2, p_3, p_4, p_5) - i \cdot s \cdot diag(1,1,1,1,0) - \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} |$$

$$|t, s \in R, A \in \mathbf{s}p(2)\}.$$

Lemma 9 is proved.

Theorem 1. *Let $\bar{p} = (p_1, p_2, p_3, p_4, p_5)$ satisfies to the following conditions*

1) $p_{\sigma(1)} + p_{\sigma(2)} - p_{\sigma(3)} - p_{\sigma(4)}$ is relatively prime with $p_{\sigma(5)}$;
2) $p_1, p_2, p_3, p_4, p_5 > 0$;
3) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} > p_{\sigma(4)} + p_{\sigma(5)}$;
4) $p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)} + p_{\sigma(4)} > 3p_{\sigma(5)}$;
5) $3(p_{\sigma(1)} + p_{\sigma(2)}) > p_{\sigma(3)} + p_{\sigma(4)} + p_{\sigma(5)}$
for every permutation $\sigma \in S_5$.

Then $M_{\bar{p}}$ is positively curved.

Proof of Theorem 1.

Let assume that the statement of Theorem is not valid. Then, by Lemma 9, there exist $g \in G$ and $X, Y \in \mathbf{u}(5)$ such that X, Y are linearly independent and orthogonal to D_g and $K(X,Y) = 0$. By Lemma 5, we may suppose that $Y \in \mathbf{k} = \mathbf{u}(4) \oplus \mathbf{u}(1)$ and

$$[X_p, Y] = [X_k, Y] = 0.$$

Let consider two possible cases.

Case 1: $X \in \mathbf{k}$. Then

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & it \end{pmatrix}, \; Y = \begin{pmatrix} Y_1 & 0 \\ 0 & is \end{pmatrix}$$

where $X_1, Y_1 \in \mathbf{u}(4), t, s \in R$. The condition that $[X,Y] = 0$ means that $[X_1, Y_1] = 0$. It follows from orthogonality to D_g that

$$\langle X, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \rangle = \langle Y, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \rangle = 0, \forall A \in \mathbf{s}p(2),$$

$$\langle X, i \cdot diag(1,1,1,1,0) \rangle = \langle Y, i \cdot diag(1,1,1,1,0) \rangle = 0,$$

$$\langle X, Ad(g^{-1}) \cdot i \cdot diag(p_1,p_2,p_3,p_4,p_5) \rangle = \langle Y, Ad(g^{-1}) \cdot i \cdot diag(p_1,p_2,p_3,p_4,p_5) \rangle = 0.$$

Since $X, Y \in \mathbf{k}$, by Lemma 4 the last equalities also valid for the metric $\langle \; \rangle_0$. Therefore X_1, Y_1 are orthogonal to $i \cdot diag(1,1,1,1)$ that means that $X_1, Y_1 \in \mathbf{s}u(4)$ and

$$\langle X_1, \mathbf{s}p(2) \rangle_0 = \langle Y_1, \mathbf{s}p(2) \rangle_0 = 0,$$

where $\mathbf{s}p(2)$ is standard embedded into $\mathbf{s}u(4)$.

It is known that $SU(4)/Sp(2)$ is a symmetric rank one space, diffeomorphic to S^5, with positively curved metric. Therefore X_1 and Y_1 are linearly dependent. Consequently, we may suppose that $X_1 = 0$ with the same $Span(X,Y)$.

Hence, we may suppose that

$$X = i \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\langle X, Ad(g^{-1}) \cdot i \cdot diag(p_1,p_2,p_3,p_4,p_5) \rangle_0 = 0$. Let consider a function

$$f_X : Ad(G) \cdot i \cdot diag(p_1,p_2,p_3,p_4,p_5) \to R : Z \mapsto \langle Z, X \rangle_0.$$

By Lemma 7, f_X attains it's extremal values at points of

$$Ad(G) \cdot i \cdot diag(p_1,p_2,p_3,p_4,p_5) \cap \{diagonal \quad matrices\}.$$

Since two adjoint diagonal matrices coincide up to permutation of their elements, the extremal values of f_X are contained in $\{p_1, p_2, p_3, p_4, p_5\} \subset (0, \infty)$ and thus we arrive at a contradiction.

Case 2: X does not lie in $\mathbf{k}$. Then

$$X_p = \begin{pmatrix} 0 & x \\ -x^* & 0 \end{pmatrix}, \; Y = \begin{pmatrix} Y_1 & 0 \\ 0 & it \end{pmatrix},$$

where $t \in R, Y_1 \in \mathbf{u}(4), x \in C^4 \setminus 0$, and $[X_p, Y] = 0$, i.e.,

$$[X_p, Y] = \begin{pmatrix} 0 & itx \\ -x^*Y_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & Y_1x \\ -itx^* & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & itx - Y_1x \\ itx^* - x^*Y_1 & 0 \end{pmatrix} = 0.$$

Consequently,

$$Y_1x = itx.$$

Since $\langle Y, i \cdot diag(1,1,1,1,0)\rangle = 0$, we have $\langle Y, i \cdot diag(1,1,1,1,0)\rangle_0 = 0$ and, therefore,

$$Y_1 \in \mathbf{s}u(4).$$

Since $Y_1x = itx$, there exists $h_1 \in SU(4)$ such that $h_1Y_1h_1^{-1} = i \cdot diag(s_1, s_2, s_3, t)$ where $s_1 + s_2 + s_3 + t = 0$. Denote

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & 1 \end{pmatrix} \in SU(5),$$

then

$$Y = h^{-1} \cdot i \cdot diag(s_1, s_2, s_3, t, t) \cdot h.$$

Let

$$H_1 = i \cdot diag(s_1, s_2, s_3, t) \in \mathbf{s}u(4), H = i \cdot diag(s_1, s_2, s_3, t, t) \in \mathbf{u}(5).$$

The following condition

$$\langle Y, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\rangle = \frac{1}{2}\langle Y, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\rangle_0 = 0$$

means that

$$\langle Y_1, A\rangle_0 = 0$$

for every $A \in \mathbf{s}p(2)$. By using of two-sided invariance of the metric $\langle \quad \rangle_0$, we obtain :

$$\langle H_1, h_1\mathbf{s}p(2)h_1^{-1}\rangle_0 = 0.$$

It immediately follows from Lemma 8 that there are two possible values of H_1 up to permutation of coordinates:

$$H_1 = i \cdot t \cdot diag(1, 1, -1, -1)$$

or

$$H_1 = i \cdot t \cdot diag(1, 1, 1, -3)$$

where $t \in R$. Therefore we may suppose that H satisfies one of three possibilities :

$$H = i \cdot diag(1, 1, 1, -1, -1),$$

$$H = i \cdot diag(1, 1, 1, 1, -3),$$

$$H = i \cdot diag(3, 3, -1, -1, -1).$$

It remains to consider a condition

$$0 = 2\langle Y, Ad(g^{-1}) \cdot i \cdot diag(p_1, p_2, p_3, p_4, p_5)\rangle =$$

$$= \langle Y, Ad(g^{-1}) \cdot i \cdot diag(p_1, p_2, p_3, p_4, p_5)\rangle_0 =$$

$$= \langle h^{-1}Hh, Ad(g^{-1}) \cdot i \cdot diag(p_1, p_2, p_3, p_4, p_5)\rangle_0 =$$

$$= \langle H, Ad(g') \cdot i \cdot diag(p_1, p_2, p_3, p_4, p_5)\rangle_0,$$

where $g' = hg^{-1} \in G$.

Let consider a function

$$f_H : Ad(G) \cdot i \cdot diag(p_1, p_2, p_3, p_4, p_5) \to R : X \mapsto \langle X, H\rangle_0.$$

By Lemma 7, it's extremal values are attained at the set

$$\{p_{i_1} + p_{i_2} + p_{i_3} - p_{i_4} - p_{i_5}, 3(p_{i_1} + p_{i_2}) - p_{i_3} - p_{i_4} - p_{i_5}, p_{i_1} + p_{i_2} + p_{i_3} + p_{i_4} - 3p_{i_5}|$$

$$|\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}\}$$

which lies in $(0, \infty)$ by the condition of Theorem. Thus we arrive at a contradiction.

Theorem 1 is proved.

It is easy to see that all conditions of Theorem 1 hold for $p_1 = 1, p_2 = p_3 = p_4 = p_5 = q^n$ where q is a prime number and n is a nonnegative integer.

4. Topology of spaces $M_{\bar{p}}$

Let denote by $\sigma_i(\bar{p})$ the i-th elementary symmetric function of p_1, p_2, p_3, p_4, p_5.

Lemma 10. *$(Sp(2)\times S^1)/\pm(E,1)$ is diffeomorphic to $Sp(2)\times S^1$.*

Proof of Lemma 10.

Let consider the mapping

$$\phi : Sp(2)\times S^1 \to Sp(2)\times S^1 : (A,z)\mapsto (A\cdot diag(z,z,\bar z,\bar z), z^2).$$

Put $\phi'(A,z)=(B,w)$. Then $z^2=w, z=\pm\sqrt{w}$ that means

$$(A,z)=\pm(B\cdot diag(\sqrt{\bar w},\sqrt{\bar w},\sqrt{w},\sqrt{w}),\sqrt{w}).$$

Thus, the mapping ϕ' induces a bijection

$$\phi : \frac{Sp(2)\times S^1}{\pm(E,1)} \to Sp(2)\times S^1$$

which evidently occurs to be a diffeomorphism.

Lemma 10 is proved.

Theorem 2. *Let $(p_{\sigma(1)}+p_{\sigma(2)}-p_{\sigma(3)}-p_{\sigma(4)})$ and $p_{\sigma(5)}$ are relatively prime for every permutation $\sigma\in S_5$. Then the space $M_{\bar p}$ is simply connected.*

Proof of Theorem 2.

Let consider the fragment of the exact homotopy sequence of the fiber bundle $\bar\pi : G\to M_{\bar p}$ with the fiber $P=S^1\times(Sp(2)\times S^1)/\pm(E,1)$:

$$\pi_1(S^1\times\frac{Sp(2)\times S^1}{\pm(E,1)}) \xrightarrow{i_*} \pi_1(U(5)) \xrightarrow{\bar\pi_*} \pi_1(M_{\bar p}) \to 0$$

where i is an embedding of P as the fiber over the unit element $E\in U(5)$. Since ϕ is a diffeomorphism, we obtain

$$\pi_1(S^1\times Sp(2)\times S^1) \xrightarrow{j_*} \pi_1(U(5)) \xrightarrow{\bar\pi_*} \pi_1(M_{\bar p}) \to 0,$$

where $j=i\circ(id\times\phi^{-1})$. Thus we have the following exact sequence

$$\mathbf{Z}\oplus\mathbf{Z} \xrightarrow{j_*} \mathbf{Z} \xrightarrow{\bar\pi_*} \pi_1(M_{\bar p}) \to 0.$$

Let compute j_*. We take in the group $\mathbf{Z}\oplus\mathbf{Z}=\pi_1(S^1\times Sp(2)\times S^1)$ it's generators $(1,0)$ and $(0,1)$ which are realized by loops

$$\xi_1(t)=(e^{2\pi it},(E,1)),\quad \xi_2(t)=(1,(E,e^{2\pi it})),\quad 0\le t\le 1.$$

We take by a generator in $\mathbf{Z}=\pi_1(U(5))$ a homotopy class given by the following winding of torus :

$$\xi(t)=diag(e^{2\pi ix_1t},e^{2\pi ix_2t},e^{2\pi ix_3t},e^{2\pi ix_4t},e^{2\pi ix_5t}), 0\le t\le 1, x_k\in Z, \sum_{k=1}^{5}x_k=1.$$

The loop ξ_1 is mapped into

$$j(\xi_1(t)) = i(e^{2\pi it}, \pm(E,1)) = diag(e^{2\pi ip_1t}, e^{2\pi ip_2t}, e^{2\pi ip_3t}, e^{2\pi ip_4t}, e^{2\pi ip_5t}),\ 0 \le t \le 1$$

and the loop ξ_2 is mapped into

$$j(\xi_2(t)) = i(1, \pm(diag(e^{-\pi it}, e^{-\pi it}, e^{\pi it}, e^{\pi it}), e^{\pi it})) = diag(1, 1, e^{2\pi it}, e^{2\pi it}, 1),\ 0 \le t \le 1.$$

We obtain that

$$j_*(1,0) = \sigma_1(\bar{p}) \cdot 1, \quad j_*(0,1) = 2 \cdot 1.$$

It follows, in particular, from conditions of Theorem that 2 and $\sigma_1(\bar{p})$ are relatively prime and, since that, we conclude that j_* is an epimorphism. It immediately follows from the exact sequence quoted above that the manifold $M_{\bar{p}}$ is simply connected.

Theorem 2 is proved.

Put $G = U(5)$ and $P = S^1 \times (Sp(2) \times S^1)/\mathbf{Z}_2 \subset G \times G$. We denote by $M = M_{\bar{p}} = G/P$ the space of orbits, and denote by $\bar{\pi} : G \to M$ the principal bundle with the structure group P. Let $\pi_G : E_G \to B_G$ and $\pi_P : E_P \to B_P$ be universal coverings for groups G and P, respectively, with contractible covering spaces E_G and E_P. Let consider the following commutative diagram :

$$\begin{array}{ccccccc} & G & \stackrel{p_2}{\longleftarrow} & E_P \times G & \stackrel{p_1}{\longrightarrow} & E_P & \\ \pi_G & \downarrow & & \downarrow & & \downarrow & \pi_P \\ & M & \stackrel{\bar{p}_2}{\longleftarrow} & G//P & \stackrel{\bar{p}_1}{\longrightarrow} & B_P & \end{array}$$

Here p_1 and p_2 are natural projections onto first and second factors, $G//P$ is the space of orbits of the natural action of P on $E_P \times G$. Since the fiber of $\bar{p}_2$ is diffeomorphic to contractible space E_P, $\bar{p}_2^*$ maps $H^*(M)$ isomorphically onto $H^*(G//P)$. Let consider the spectral sequence of the fiber bundle $p = \bar{p}_1 : G//P \to B_P$ with the fiber G. Since $H^*(G)$ is torsion free, the initial term $E_2 = H^*(B_P) \otimes H^*(G)$. The term E_∞ is attached to $H^*(M)$. Let compute differentials of this spectral sequence.

We consider the diagram:

$$\begin{array}{ccccccc} G//P = & (E_{G^2} \times G)/P & \stackrel{\hat{\rho}}{\longrightarrow} & (E_{G^2} \times G)/G^2 & \stackrel{f}{\longleftarrow} & B_G & = E_{G^2}/\delta G \\ & p \downarrow & & \downarrow p' & & \downarrow \triangle & \\ & B_P & \stackrel{\rho}{\longrightarrow} & B_{G^2} & \stackrel{id}{\longleftarrow} & B_{G^2} & \end{array}$$

Here we put $E_P = E_{G^2}$, $B_P = E_{G^2}/P$, and $B_{G^2} = E_{G^2}/G^2$. We denote here by $\rho : B_P \to B_{G^2}$ and $\triangle : B_G \to B_{G^2}$ natural projections. We also denote by $\delta : G \to G^2$ the diagonal embedding and denote by $\hat{\rho}$ the fibered mapping whose restrictions onto fibers are homeomorphisms. The mapping $f : (\delta G)e \mapsto G^2(e,1)$ is an isomorphism of fiber bundles.

Let compute differentials of the spectral sequence of the fiber bundle $\triangle$. We identify the ring of cohomologies $H^*(G)$ with an interior algebra with generators

$z_1, z_3, \dots, z_9$. Then we put $H^*(B_G) = \mathbf{Z}[\bar{z}_1, \bar{z}_3, \dots, \bar{z}_9]$ where $\bar{z}_i$ s the image of z_i under transgression of the fiber bundle π_G. Since $B_{G^2} = B_G \times B_G$,

$$H^*(B_{G^2}) = H^*(B_G) \otimes H^*(B_G) = \mathbf{Z}[\bar{x}_1, \bar{y}_1, \bar{x}_3, \bar{y}_3, \dots, \bar{x}_9, \bar{y}_9]$$

where $\bar{x}_i = \bar{z}_i \otimes 1, \bar{y}_i = 1 \otimes \bar{z}_i$.

The initial term E_2 is isomorphic to $H^*(B_{G^2}) \otimes H^*(G)$. Let denote by $k_i : H^*(B_{G^2}) \to E_i^{*,0}$ the natural projection. Then it is well-known that

$$\triangle^* = k_\infty : H^*(B_{G^2}) \to E_\infty^{*,0} \subset H^* B_G.$$

Lemma 11.

$$1)\ d_j(1 \otimes z_i) = 0, \quad j \leq i,\ i = 3, 5, 7;$$

$$2)\ d_{i+1}(1 \otimes z_i) = \pm k_{i+1}(\bar{x}_i - \bar{y}_i),\ i = 1, 3, 5, 7.$$

Proof of Lemma 11.

Let consider the term E_2 of the spectral sequence of the fiber bundle $B_G \to B_{G^2}$.

z_7	0	$*$	0	$*$	0	$*$
$z_1 z_5$	0	$z_1 z_5 \bar{x}_1, z_1 z_5 \bar{y}_1$	0	$*$	0	$*$
z_5	0	$*$	0	$*$	0	$*$
$z_1 z_3$	0	$z_1 z_3 \bar{x}_1, z_1 z_3 \bar{y}_1$	0	$z_1 z_3 \otimes E_2^{4,0}$	0	$*$
z_3	0	$z_3 \bar{x}_1, z_3 \bar{y}_1$	0	$*$	0	$*$
0	0	0	0	0	0	0
z_1	0	$z_1 \bar{x}_1, z_1 \bar{y}_1$	0	$z_1 \bar{x}_1^2, z_1 \bar{x}_1 \bar{y}_1, z_1 \bar{y}_1^2$ $z_1 \bar{x}_3, z_1 \bar{y}_3$	0	$*$
1	0	$\bar{x}_1, \bar{y}_1$	0	$\bar{x}_1^2, \bar{x}_1 \bar{y}_1, \bar{y}_1^2$ $\bar{x}_3, \bar{y}_3$	0	$\bar{x}_1^3, \bar{x}_1^2 \bar{y}_1, \bar{x}_1 \bar{y}_1^2, \bar{y}_1^3$ $\bar{x}_1 \bar{x}_3, \bar{x}_1 \bar{y}_3, \bar{y}_1 \bar{x}_3, \bar{y}_1 \bar{y}_3$ $\bar{x}_5, \bar{y}_5$

We have

$$\triangle^*(1 \otimes u) = \triangle^*(u \otimes 1) = u.$$

for every $u \in H^*(B_G)$. Since the kernel of $\triangle^2$ coincides with $d_2(\mathbf{Z}(z_1))$, $d_2(\mathbf{Z}(z_1)) = \mathbf{Z}(\bar{x}_1 - \bar{y}_1)$. We conclude that

$$d_2(z_1) = \pm(\bar{x}_1 - \bar{y}_1) = \pm k_2(\bar{x}_1 - \bar{y}_1).$$

Thus, (2) is proved for $i + 1$.

Then we have

$$d_2(z_1 z_3 \bar{x}_1) = z_3 \bar{x}_1^2 - z_3 \bar{x}_1 \bar{y}_1,$$

$$d_2(z_1 z_3 \bar{y}_1) = z_3 \bar{x}_1 \bar{y}_1 - z_3 \bar{y}_1^2.$$

Therefore, $Ker(d_2^{2,4}) = 0$ and we get that $d_2^{0,5} = 0$. Analogously, we conclude that $Ker(d_2^{2,6}) = 0, Ker(d_2^{4,4}) = 0$ and, therefore, $d_2^{0,7} = d_4^{0,7} = 0$. Triviality of

other differentials follows from dimensional reasons. Thus, it is left to prove (2) for $i = 3, 5, 7$. One can easily see that

$$Ker\triangle^4 = \mathbf{Z}(\bar{x}_3 - \bar{y}_3) \oplus \mathbf{Z}(\bar{x}_1^2 - \bar{x}_1\bar{y}_1, \bar{x}_1\bar{y}_1 - \bar{y}_1^2),$$

and, from other side,

$$Ker\triangle^4 = Im(d_2^{2,1}) \oplus Im(d_4^{0,3}).$$

Taking into account that

$$Im(d_2^{2,1}) = \mathbf{Z}(\bar{x}_1^2 - \bar{x}_1\bar{y}_1, \bar{x}_1\bar{y}_1 - \bar{y}_1^2),$$

we derive that

$$d_4(z_3) = \pm k_4(\bar{x}_3 - \bar{y}_3).$$

In the same manner we obtain that

$$Ker\triangle^6 = \mathbf{Z}(\bar{x}_5 - \bar{y}_5) \oplus \mathbf{Z}(\bar{x}_1^3 - \bar{x}_1^2\bar{y}_1, \bar{x}_1^2\bar{y}_1 - \bar{x}_1\bar{y}_1^2, \bar{x}_1\bar{y}_1^2 - \bar{y}_1^3)\oplus$$
$$\oplus\mathbf{Z}(\bar{x}_1\bar{x}_3 - \bar{y}_1\bar{x}_3, \bar{x}_1\bar{y}_3 - \bar{y}_1\bar{y}_3, \bar{x}_1\bar{x}_3 - \bar{x}_1\bar{y}_3),$$

and

$$Ker\triangle^6 = Im(d_2^{4,1}) \oplus Im(d_4^{2,4}) \oplus Im(d_6^{0,5}).$$

Taking into account that we proved before that first two summands from the last expression coincides respectively with the same from the preceding one we get

$$d_6(z_5) = \pm k_6(\bar{x}_5 - \bar{y}_5).$$

Analogously one can prove that $d_8(z_7) = \pm k_8(\bar{x}_7 - \bar{y}_7)$.

Lemma 11 is proved.

Lemma 12. *Let $d_j, j \geq 1$ be differentials in the spectral sequence of the fiber bundle $p : G//P \to B_P$. Then*

1) $d_j(1 \otimes z_i) = 0, \quad j \leq i, \ i = 3, 5, 7$;

2) $d_{i+1}(1 \otimes z_i) = \pm k_{i+1}\rho^*(\bar{x}_i - \bar{y}_i), \ i = 1, 3, 5, 7$

where $\rho : B_P \to B_{G^2}$ is induced by the embedding $P \subset G^2$.

Proof of Lemma 12.

Let consider the second diagram. The fibered mapping $(\hat{\rho}, \rho)$ generates the homomorphism $\rho^\sharp$, of spectral sequences, and, moreover, $\rho_2^\sharp = \rho^* \otimes i : H^*B_{G^2} \otimes H^*G \to H^*B_P \otimes H^*G$ where i is an isomorphism. We put $i(1 \otimes z_i) = 1 \otimes z_i$. Then the following identities

$$d_j(1 \otimes z_i) = \rho^\sharp(d'_j(1 \otimes z_i)) = \rho^\sharp(0) = 0, j \leq i,$$

$$d_{i+1}(1 \otimes z_i) = \rho^\sharp(d'_{i+1}(1 \otimes z_i)) = \pm\rho^\sharp(k'_{i+1}(\bar{x}_i - \bar{y}_i)) = \pm k_{i+1}(\rho^*(\bar{x}_i - \bar{y}_i)$$

hold.

Lemma 12 is proved.

Let G be a Lie group and T^n be a maximal torus in G where $i : T^n \to G$ is an embedding and $j : B_{T^n} \to B_G$ is a natural projection. We denote by $a_1, \ldots, a_n$ generators of H^1T^n. Then we have $H^*B_{T^n} = \mathbf{Z}[\bar{a}_1, \ldots, \bar{a}_n]$. Let denote by I_G the algebra of polynoms in $H^*B_{T^n}$ that are invariant under the action of the Weyl group $W(G)$.

Borel Theorem. *([Bo]) Let H^*G and $H^*(G/T^n)$ are torsion free. Then j^* : $H^*B_G \to H^*B_{T^n}$ is a monomorphism and it's image coincides with I_G.*

Borel proved ([Bo]) that conditions of this theorem hold for every classic group. We have $G = U(5)$, $\bar{z}_1 = \sigma_1(\bar{d}_1, \dots, \bar{d}_5), z_3 = \sigma_2(\bar{d}_1, \dots, \bar{d}_5), \dots, z_9 = \sigma_5(\bar{d}_1, \dots, \bar{d}_5)$ where $d_1, \dots, d_5$ are cocycles which are adjoint to cycles $D_1, \dots, D_5$ that are defined by $D_i(t) = diag(1, \dots, e^{2\pi i t}, \dots, 1), 0 \le t \le 1$.

Let consider $P \subset G^2$:

$$P = S^1 \times \frac{Sp(2) \times S^1}{\mathbf{Z}_2} \simeq S^1 \times S^1 \times Sp(2),$$

The ring of cohomologies H^*P is torsion free. Next, let T^3 be a maximal torus in $Sp(2) \times S^1$. Then $T^3/\mathbf{Z}_2$ is a maximal torus in $(Sp(2) \times S^1)/\mathbf{Z}_2$, i.e.,

$$\frac{S^1 \times Sp(2)}{\mathbf{Z}_2} / \frac{T^3}{\mathbf{Z}_2} \simeq \frac{S^1 \times Sp(2)}{T^3}.$$

One can see now that for P the conditions of Borel Theorem hold. Let T be a torus in G^2 and S be a torus in P. Denote by $i : S \to T$ an embedding and denote by $j : B_S \to B_T$ a natural projection. Let consider the following diagram

$$\begin{array}{ccccc} & H^*B_S & \overset{j^*}{\longleftarrow} & H^*B_T & \\ j_S^* & \uparrow & & \uparrow & j_T^* \\ & H^*B_P & \overset{\rho^*}{\longleftarrow} & H^*B_{G^2} & \end{array}$$

Let choose a basis $A_1, \dots, A_5, B_1, \dots, B_5$ of cycles in $H_1(T)$ as follows :

$$A_i(t) = (1, diag(1, \dots, e^{2\pi i t}, \dots, 1)), 0 \le t \le 1,$$

$$B_i(t) = (diag(1, \dots, e^{2\pi i t}, \dots, 1), 1), 0 \le t \le 1,$$

and denote by $a_1, \dots, a_5, b_1, \dots, b_5$ cocycles ($\in H^1(T)$) that are adjoint to elements of this basis. Let choose a basis $C_1, \dots, C_4$ of cycles in $H_1(S)$ as follows :

$$C_1(t) = (e^{2\pi i t}, \pm(E, 1)),$$
$$C_2(t) = (1, \pm(diag(e^{\pi i t}, e^{\pi i t}, e^{-\pi i t}, e^{-\pi i t}), e^{\pi i t})),$$
$$C_3(t) = (1, \pm(diag(e^{2\pi i t}, 1, e^{-2\pi i t}, 1), 1)),$$
$$C_4(t) = (1, \pm(diag(1, e^{2\pi i t}, 1, e^{-2\pi i t}), 1)),$$
$$0 \le t \le 1,$$

and denote by c_1, c_2, c_3, c_4 cocycles ($H^1(S)$) that are adjoint to elements of this basis. Then we have

$$i_*(C_1) = p_1 B_1 + p_2 B_2 + p_3 B_3 + p_4 B_4 + p_5 B_5,$$

$$i_*(C_2) = A_1 + A_2, \quad i_*(C_3) = A_1 - A_3, \quad i_*(C_4) = A_2 - A_4.$$

Consequently,

$$i^*(a_1) = c_2 + c_3, \quad i^*(a_2) = c_2 + c_4, \quad i^*(a_3) = -c_3, \quad i^*(a_4) = -c_4, \quad i^*(a_5) = 0,$$

$$i^*(b_1) = p_1 c_1, \quad i^*(b_2) = p_2 c_1, \quad i^*(b_3) = p_3 c_1, \quad i^*(b_4) = p_4 c_1, \quad i^*(b_5) = p_5 c_1.$$

Since transgression is natural, we have

$$j^*(\bar{a}_i) = \overline{i^*(a_i)}, \quad j^*(\bar{b}_i) = \overline{i^*(b_i)}.$$

By the diagram given above, ρ^* is the restriction of j^* onto I_{G^2}.

We identify $H^*B_{G^2}$ with the subalgebra, of H^*B_T, generated by

$$\sigma_i(\bar{a}_1, \dots, \bar{a}_5), \quad \sigma_i(\bar{b}_1, \dots, \bar{b}_5), \quad i = 1, 2, \dots, 5.$$

We identify the ring of cohomologies H^*B_P with subalgebra, in H^*B_S, that is invariant under $W(P)$. Notice that in our definitions

$$a_i = 1 \otimes d_i, \; b_i = d_i \otimes 1.$$

Let compute $W(P)$. Elements of $W(P)$ are induced by elements of $W(S^1 \times S^1 \times Sp(2))$. Hence, generators ϕ_1, ϕ_2, ϕ_3, of $W(P)$, act on homologies of S as follows

$$\begin{array}{llll}
\phi_1: & C_1 \mapsto C_1 & \phi_2: \quad C_1 \mapsto C_1 & \phi_3: \quad C_1 \mapsto C_1 \\
 & C_2 \mapsto C_2 - C_3 & \quad\;\; C_2 \mapsto C_2 - C_4 & \quad\;\; C_2 \mapsto C_2 \\
 & C_3 \mapsto -C_3 & \quad\;\; C_3 \mapsto C_3 & \quad\;\; C_3 \mapsto C_4 \\
 & C_4 \mapsto C_4 & \quad\;\; C_4 \mapsto -C_4 & \quad\;\; C_4 \mapsto C_3,
\end{array}$$

that means that their action on cohomologies is given by

$$\begin{array}{llll}
\phi_1: & c_1 \mapsto c_1 & \phi_2: \quad c_1 \mapsto c_1 & \phi_3: \quad c_1 \mapsto c_1 \\
 & c_2 \mapsto c_2 & \quad\;\; c_2 \mapsto c_2 & \quad\;\; c_2 \mapsto c_2 \\
 & c_3 \mapsto -c_2 - c_3 & \quad\;\; c_3 \mapsto c_3 & \quad\;\; c_3 \mapsto c_4 \\
 & c_4 \mapsto c_4 & \quad\;\; c_4 \mapsto -c_2 - c_4 & \quad\;\; c_4 \mapsto c_3.
\end{array}$$

Thus H^*B_P is a subalgebra of $\mathbf{Z}[\bar{c}_1, \bar{c}_2]$ that is invariant under $W(P)$. Let find multiplicative generators of H^*B_P.

Lemma 13. *Let denote*

$$\bar{f} = (\bar{c}_3^2 + \bar{c}_4^2) + \bar{c}_2(\bar{c}_3 + \bar{c}_4),$$

$$\bar{g} = \bar{c}_3^2 \bar{c}_4^2 + \bar{c}_2 \bar{c}_3 \bar{c}_4 (\bar{c}_3 + \bar{c}_4) + \bar{c}_2^2 \bar{c}_3 \bar{c}_4.$$

Then $H^*B_P = \mathbf{Z}[\bar{c}_1, \bar{c}_2, \bar{f}, \bar{g}]$.

Proof of Lemma 13.

Let consider the natural embedding $H^*B_S = \mathbf{Z}[\bar c_1, \bar c_2, \bar c_3, \bar c_4] \subset \mathbf{R}[\bar c_1, \bar c_2, \bar c_3, \bar c_4]$. We denote by $A_{\mathbf{R}}$ he subalgebra of $\mathbf{R}[\bar c_1, \bar c_2, \bar c_3, \bar c_4]$ that is invariant under $W(P)$. Then we have $H^*B_P = A_{\mathbf{Z}} = A_{\mathbf{R}} \cap \mathbf{Z}[\bar c_1, \bar c_2, \bar c_3, \bar c_4]$.

We define an isomorphism $\tau : \mathbf{R}[x_1, x_2, x_3, x_4] \to \mathbf{R}[\bar c_1, \bar c_2, \bar c_3, \bar c_4]$ as follows:

$$\begin{aligned} x_1 &\mapsto \bar c_1 \\ x_2 &\mapsto \bar c_2 \\ x_3 &\mapsto \bar c_3 + \tfrac{1}{2}\bar c_2 \\ x_4 &\mapsto \bar c_4 + \tfrac{1}{2}\bar c_2. \end{aligned}$$

The $W(P)$ is conjugated by τ to the group W' that acts on $\mathbf{R}[x_1, x_2, x_3, x_4]$ and generated by the following elements ;

$$\begin{array}{llllll} \phi_1': & x_1 \mapsto x_1 & \phi_2': & x_1 \mapsto x_1 & \phi_3': & x_1 \mapsto x_1 \\ & x_2 \mapsto x_2 & & x_2 \mapsto x_2 & & x_2 \mapsto x_2 \\ & x_3 \mapsto x_4 & & x_3 \mapsto -x_3 & & x_3 \mapsto x_3 \\ & x_4 \mapsto x_3 & & x_4 \mapsto x_4 & & x_4 \mapsto -x_4. \end{array}$$

Thus, we obtain that W' is the Weyl group of group $S^1 \times S^1 \times Sp(2)$ and it is absolutely evident that the subalgebra of $\mathbf{R}[x_1, x_2, x_3, x_4]$ that is invariant under W' coincides with $A'_{\mathbf{R}} = \mathbf{R}[x_1, x_2, x_3^2 + x_4^2, x_3^2 x_4^2]$. Hence, $A_{\mathbf{R}} = \mathbf{R}[\tau(x_1), \tau(x_2), \tau(x_3^2 + x_4^2), \tau(x_3^2 x_4^2)]$. Nondifficult computations give

$$\tau(x_1) = \bar c_1, \tau(x_2) = \bar c_2,$$

$$\tau(x_3^2 + x_4^2) = (\bar c_3^2 + \bar c_4^2) + \bar c_2(\bar c_3 + \bar c_4) + \frac{1}{2}\bar c_2^2,$$

$$\tau(x_3^2 x_4^2) = \bar c_3^2 \bar c_4^2 + \bar c_2 \bar c_3 \bar c_4(\bar c_3 + \bar c_4) + \bar c_2^2 \bar c_3 \bar c_4 + + \frac{1}{4}\bar c_2^2((\bar c_3^2 + \bar c_4^2) + \bar c_2(\bar c_3 + \bar c_4) + \frac{1}{4}\bar c_2^2).$$

Thus, we have $A_{\mathbf{R}} = \mathbf{R}[\bar c_1, \bar c_2, (\bar c_3^2 + \bar c_4^2) + \bar c_2(\bar c_3 + \bar c_4), \bar c_3^2 \bar c_4^2 + \bar c_2 \bar c_3 \bar c_4(\bar c_3 + \bar c_4) + \bar c_2^2 \bar c_3 \bar c_4] = \mathbf{R}[\bar c_1, \bar c_2, \bar f, \bar g]$. Since generators of $A_{\mathbf{R}}$ lie in $\mathbf{Z}[\bar c_1, \bar c_2, \bar c_3, \bar c_4]$, one get

$$A_{\mathbf{Z}} = \mathbf{Z}[\bar c_1, \bar c_2, \bar f, \bar g].$$

Lemma 13 is proved.

Theorem 3. *The space $M_{\bar p}$ has the following groups of cohomologies*

$$1) H^i = \begin{cases} \mathbf{Z}, & for \quad i=0,2,4,9,11,13, \\ 0, & for \quad i=1,3,5,7,10,12; \end{cases}$$

2) the groups $H^6(M_{\bar p})$ and $H^8(M_{\bar p})$ are finite and their orders are equal to

$$r = |\sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3|.$$

Proof of Theorem 3.

We denote $\sigma_i = \sigma_i(p_1, \dots, p_5)$. Let consider the term E_2 of the spectral sequence of the fiber map $G//P \to B_P$.

z_7	0	$*$	0	$*$	0	$*$	0	$*$
z_1z_5	0	$z_1z_5\bar{c}_1, z_1z_5\bar{c}_2$	0	$*$	0	$*$	0	$*$
z_5	0	$z_5\bar{c}_1, z_5\bar{c}_2$	0	$*$	0	$*$	0	$*$
z_1z_3	0	$z_1z_3\bar{c}_1, z_1z_3\bar{c}_2$	0	$z_1z_3\bar{c}_1^2, z_1z_3\bar{c}_1\bar{c}_2$ $z_1z_3\bar{c}_2^2, z_1z_3\bar{f}$	0	$*$	0	$*$
z_3	0	$z_3\bar{c}_1, z_3\bar{c}_2$	0	$z_3\bar{c}_1^2 z_3\bar{c}_1\bar{c}_2$ $z_3\bar{c}_2^2, z_3\bar{f}$	0	$*$	0	$*$
0	0	0	0	0	0	0	0	0
z_1	0	$z_1\bar{c}_1, z_1\bar{c}_2$	0	$z_1\bar{c}_1^2, z_1\bar{c}_1\bar{c}_2$ $z_1\bar{c}_2^2, z_1\bar{f}$	0	$z_1\bar{c}_1^3, z_1\bar{c}_1^2\bar{c}_2, z_1\bar{c}_1\bar{c}_2^2$ $z_1\bar{c}_2^3, z_1\bar{c}_1\bar{f}, z_1\bar{c}_2\bar{f}$	0	$*$
1	0	$\bar{c}_1, \bar{c}_2$	0	$\bar{c}_1^2, \bar{c}_1\bar{c}_2$ $\bar{c}_2^2, \bar{f}$	0	$\bar{c}_1^3, \bar{c}_1^2\bar{c}_2, \bar{c}_1\bar{c}_2^2$ $\bar{c}_2^3, \bar{c}_1\bar{f}, \bar{c}_2\bar{f}$	0	$\bar{c}_1^3\bar{c}_2, \bar{c}_1^2\bar{c}_2^2, \bar{c}_1\bar{c}_2^3$ $\bar{c}_2^4, \bar{c}_1^2\bar{f}, \bar{c}_1\bar{c}_2\bar{f}$ $\bar{c}_2^2\bar{f}, \bar{f}^2, \bar{g}, \bar{c}_1^4$

We have

$$d_2 z_1 = \pm\rho^*(\bar{x}_1 - \bar{y}_1) = \rho^*(\sigma_1(\bar{b}_1, \dots, \bar{b}_5) - \sigma_1(\bar{a}_1, \dots, \bar{a}_5)) =$$

$$= \sigma_1 \cdot \bar{c}_1 - 2 \cdot \bar{c}_2.$$

Take $n \in \mathbf{Z}$ such that $\sigma_1 + 2n = 1$. Then

$$\frac{\mathbf{Z}(\bar{c}_1, \bar{c}_2)}{\mathbf{Z}(\sigma_1 \cdot \bar{c}_1 - 2 \cdot \bar{c}_2)} = \mathbf{Z}(n \cdot \bar{c}_1 + \bar{c}_2).$$

Denote by F_2 the image of $d_2^{2,1}$. Notice that F_2 is generated by elements

$$d_2(z_1\bar{c}_1) = \sigma_1\bar{c}_1^2 - 2\bar{c}_1\bar{c}_2, d_2(z_1\bar{c}_2) = \sigma_1\bar{c}_1\bar{c}_2 - 2\bar{c}_2^2.$$

Then we have $\mathbf{Z}(\bar{c}_1^2, \bar{c}_1\bar{c}_2, \bar{c}_2^2, \bar{f})/F_2 = \mathbf{Z}((n-n^2)\bar{c}_1^2 + \bar{c}_2^2, \bar{f})$. Denote by F_4 the image of $d_2^{4,1}$. We see that F_4 is generated by elements

$$d_2(z_1\bar{c}_1^2) = \sigma_1 \cdot \bar{c}_1^3 - 2 \cdot \bar{c}_1^2\bar{c}_2, \quad d_2(z_1\bar{c}_1\bar{c}_2) = \sigma_1 \cdot \bar{c}_1^2\bar{c}_2 - 2 \cdot \bar{c}_1\bar{c}_2^2,$$

$$d_2(z_1\bar{c}_2^2) = \sigma_1 \cdot \bar{c}_1\bar{c}_2^2 - 2 \cdot \bar{c}_2^3, \quad d_2(z_1\bar{f}) = \sigma_1 \cdot \bar{c}_1\bar{f} - 2 \cdot \bar{c}_2\bar{f}$$

Finally, denote by F_6 the image of $d_2^{6,1}$ and notice that it is generated by elements

$$\begin{aligned} d_2(z_1\bar{c}_1^3) &= \sigma_1\bar{c}_1^4 - 2\bar{c}_1^3\bar{c}_2, & d_2(z_1\bar{c}_2^3) &= \sigma_1\bar{c}_1\bar{c}_2^3 - 2\bar{c}_2^4, \\ d_2(z_1\bar{c}_1^2\bar{c}_2) &= \sigma_1\bar{c}_1^3\bar{c}_2 - 2\bar{c}_1^2\bar{c}_2^2, & d_2(z_1\bar{c}_1\bar{f}) &= \sigma_1\bar{c}_1^2\bar{f} - 2\bar{c}_1\bar{c}_2\bar{f}, \\ d_2(z_1\bar{c}_1\bar{c}_2^2) &= \sigma_1\bar{c}_1^2\bar{c}_2^2 - 2\bar{c}_1\bar{c}_2^3, & d_2(z_1\bar{c}_2\bar{f}) &= \sigma_1\bar{c}_1\bar{c}_2\bar{f} - 2\bar{c}_2^2\bar{f}. \end{aligned}$$

Let proceed to the term $E_3 = E_4$.

$$
\begin{array}{ccccccccc}
z_7 & 0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & * & 0 & * & 0 & * \\
z_5 & 0 & nz_5\bar{c}_1 + z_5\bar{c}_2 & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
z_3 & 0 & nz_3\bar{c}_1 + z_3\bar{c}_2 & 0 & \mathbf{Z}^4/z_3F_2 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
1 & 0 & Z & 0 & \mathbf{Z}^4/F_2 & 0 & \mathbf{Z}^6/F_4 & 0 & \mathbf{Z}^{10}/F_6
\end{array}
$$

We have

$$d_4(z_3) = \rho^*(\bar{x}_3 - \bar{y}_3) = \rho^*(\sigma_2(\bar{b}_1, \dots, \bar{b}_5) - \sigma_2(\bar{a}_1, \dots, \bar{a}_5)) =$$

$$= \sigma_2 \cdot \bar{c}_1^2 - \sigma_2(\bar{c}_2 + \bar{c}_3, \bar{c}_2 + \bar{c}_4, -\bar{c}_3, -\bar{c}_4) = \sigma_2 \cdot \bar{c}_1^2 - \bar{c}_2^2 + \bar{c}_3^2 + \bar{c}_4^2 + (\bar{c}_3 + \bar{c}_4)\bar{c}_2 =$$

$$= \sigma_2 \cdot \bar{c}_1^2 - \bar{c}_2^2 + \bar{f}.$$

Thus, we conclude that $\mathbf{Z}^4/(F_2 \oplus \mathbf{Z}(d_4(z_3))) = \mathbf{Z}(\bar{f}, (n-n^2)\bar{c}_1^2 + \bar{c}_2^2)/\mathbf{Z}(d_4(z_3)) = \mathbf{Z}((n-n^2)\bar{c}_1^2 + \bar{c}_2^2)$. Now we deduce that

$$d_4(nz_3\bar{c}_1 + z_3\bar{c}_2) = (\sigma_2 \cdot \bar{c}_1^2 - \bar{c}_2^2 + \bar{f})(n\bar{c}_1 + \bar{c}_2) =$$

$$= n\sigma_2 \cdot \bar{c}_1^3 + \sigma_2 \cdot \bar{c}_1^2\bar{c}_2 - n \cdot \bar{c}_1\bar{c}_2^2 - \bar{c}_2^3 + n \cdot \bar{c}_1\bar{f} + \bar{c}_2\bar{f}.$$

One can see that the last element does not vanish in $\mathbf{Z}^6/F_4$. Denote by F_1 the subgroup, of H^6B_P, generated by this this element. Denote by F_2' the image of $d_4^{4,3}$ and notice that F_2' is generated by elements $d_4(z_3\bar{f})$ and $d_4((n-n^2)z_3\bar{c}_1^2 + z_3\bar{c}_2^2)$. Nondifficult computations show that $Ker(d_4^{4,3}) = 0$. Let consider $E_5 = E_6$.

$$
\begin{array}{ccccccccc}
z_7 & 0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & * & 0 & * & 0 & * \\
z_5 & 0 & nz_5\bar{c}_1 + z_5\bar{c}_2 & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
1 & 0 & \mathbf{Z} & 0 & \mathbf{Z} & 0 & \mathbf{Z}^6/(F_4 \oplus F_1) & 0 & \mathbf{Z}^{10}/(F_6 \oplus F_2')
\end{array}
$$

We have

$$d_6(z_5) = \rho^*(\bar{x}_5 - \bar{y}_5) = \rho^*(\sigma_3(\bar{b}_1, \dots, \bar{b}_5) - \sigma_3(\bar{a}_1, \dots, \bar{a}_5)) =$$

$$= \sigma_3 \cdot \bar{c}_1^3 - \sigma_3(\bar{c}_2 + \bar{c}_3, \bar{c}_2 + \bar{c}_4, -\bar{c}_3, -\bar{c}_4) =$$

$$= \sigma_3 \cdot \bar{c}_1^3 + (\bar{c}_2 + \bar{c}_3)(\bar{c}_2 + \bar{c}_4)\bar{c}_3 + (\bar{c}_2 + \bar{c}_3)(\bar{c}_2 + \bar{c}_4)\bar{c}_4 - (\bar{c}_2 + \bar{c}_3)\bar{c}_3\bar{c}_4 - (\bar{c}_2 + \bar{c}_4)\bar{c}_3\bar{c}_4 =$$

$$= \sigma_3 \cdot \bar{c}_1^3 + (\bar{c}_3 + \bar{c}_4)\bar{c}_2^2 + (\bar{c}_3^2 + \bar{c}_4^2)\bar{c}_2 = \sigma_3 \cdot \bar{c}_1^3 + \bar{f}\bar{c}_2.$$

Let the element $d_6(z_5)$ generates the subgroup F_1' in $H^6 B_P$. In addition, denote by F_1'' the subgroup generated by the element $d_6(nz_5\bar{c}_1 + z_5\bar{c}_2)$. We deduce by simple computation which we omit that subgroups F_1' and F_1'' are nontrivial in $\mathbf{Z}^6/(F_4 \oplus F_1)$ and $\mathbf{Z}^{10}/(F_6 \oplus F_2')$, respectively. Next, we consider the term E_7.

$$\begin{array}{cccccccccc}
z_7 & 0 & * & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & * & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
1 & 0 & \mathbf{Z} & 0 & \mathbf{Z} & 0 & \mathbf{Z}^6/(F_4 \oplus F_1 \oplus F_1') & 0 & \mathbf{Z}/(F_6 \oplus F_2' \oplus F_1'')
\end{array}$$

Since $d_8(z_7) = \sigma_4\bar{c}_1^4 - \bar{g}$, the element z_7 does not survive in the next dimensions. Thus we obtain that

$$H^1 = H^3 = H^5 = H^7 = 0, H^2 = H^4 = \mathbf{Z},$$

$$H^6(M_{\bar{p}}) = \frac{\mathbf{Z}^6}{F_4 \oplus F_1 \oplus F_1'}.$$

Let now find r which is equal to the order of group H^6:

$$r = \left| det \begin{pmatrix} \sigma_1 & -2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1 & -2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 & -2 \\ n\sigma_2 & \sigma_2 & -n & -1 & n & 1 \\ \sigma_3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right|.$$

We remind that $n = (1 - \sigma_1)/2$. By Nondifficult computations, which we omit here, we obtain that

$$r = |\sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3|.$$

Theorem 3 is proved.

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